Thermohaline circulation in a two-layer model with sloping boundaries and a mid-ocean ridge

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Abstract

A model ocean basin has continental slopes at the west and north and a mid-ocean ridge running north-south. The first problem treated here is Stommel’s (1958) abyssal flow of a homogeneous fluid in the presence of topography with a source of fluid at the northwest corner. The flow is driven by uniform upwelling everywhere except over the non-flat topography where the fluid is not driven and inviscid. The solution over the topography is determined by conservation of potential vorticity with the flow being driven by matching to the known solution over the flat bottom. A second problem is the two-layer case for which we use an analysis by Salmon (1992) in which he obtained expressions for the potential vorticity distribution for inviscid, non-forced flow over topography. We produce analytical solutions for the two-layer case. Conservation of potential vorticity over the slopes leads to flows that make large north-south excursions as the fluid crosses the topographic regions over the western slope and the midocean ridge. We justify our procedure by showing a similarity between a wind-driven double gyre solution of Salmon’s that shows a cyclonic circulation on the onshore (Gulf Stream) side of the anticyclonic wind gyre and an observed cyclonic gyre in the Slope Sea inshore of the Gulf Stream. For the two-layer case we obtain flow over the midocean ridge in the lower layer that compares favorably with the excursion of particles over the mid-Atlantic Ridge as reported by Defant (1941, 1961). Our inviscid solutions contain sharp discontinuities which will have to be smoothed by friction when viscosity is added, but it is possible that friction will take the flow far from the one that we have derived. A numerical solution of the problem is planned to test this possibility.

Key words: Ocean Circulation, Thermohaline, Bottom topography effects, Abyssal circulation, Model
1 Introduction

Henry Stommel’s (1958) abyssal circulation article contained the first theoretical analysis of the deep thermohaline circulation. In its simplest form Stommel’s reasoning was that incoming heat flux via the sun’s radiation is stirred downward by wind and thermal convection and heats up the waters down to the thermocline and that this subsurface source of heat must be offset by a source of cold if the ocean is not to become continuously warmer. He envisaged an upwelling of cold, deep water as the cold source. This upwelling involves vortex stretching of columns of fluid in the lower layer and leads to a poleward meridional flow everywhere in that layer outside of western boundary layers. His analysis of the deep flow, with boundary layers to provide mass balance and sources of cold water in the polar regions to supply the upwelling water, is elegantly simple. The resulting flow pattern is still used as the zero-order circulation of the deep oceans of the world. Stommel and Arons (1960) filled in the details that Stommel had omitted.

Some years later Veronis (1976) completed Stommel’s model by adding the analytical details in the upper layer of a two-layer model in which the lower layer contains Stommel’s abyssal circulation.

A dozen years after that Kawase (1987) added a new element to the study by introducing a thermodynamically based mechanism for the upwelling. He used an earlier result of Tziperman’s (1986) in which the vertical (upwelling) velocity is evaluated from the balance between upward convection and diffusive convergence of density flux. His crude finite differencing across the interface effectively amounts to an assumption of constant gradients of density in the upper and lower layers and it results in setting \( w = \lambda h \), where \( w \) is upwelling velocity, \( \lambda \) is a constant determined by mixing across the interface and \( h \) is the departure of the interface from its equilibrium position. For small values of \( \lambda \) Kawase’s solution is the same as Stommel’s. For moderate \( \lambda \) the interior meridional flow is not uniform as in Stommel’s model but the solutions are otherwise similar. For large \( \lambda \) all of the action is in the boundary layers, western, equatorial and eastern – there is no upwelling and therefore no meridional flow in the interior. Thus, the behavior in this case is qualitatively different from that in Stommel’s model. Kawase confirmed his elegant analytical solutions with numerical simulations in which the boundary flows in the case of large \( \lambda \) are initiated by Kelvin waves travelling cyclonically around the basin.

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Since Kawase’s paper appeared, a number of investigators have examined different aspects of abyssal and thermohaline circulations. Those that have dealt with topography include studies by Hautala and Riser (1989) who included wind as well as thermally forced motions in their integration along characteristics in the ocean interior. Straub and Rhines (1990) and Kawase and Straub (1991) showed closed geostrophic contours induced by a seamount or depression. Integration along the characteristics of the inviscid equations was also used by Straub et al. (1993) to study a basin with a smoothly varying bottom and by Spall (2001) to study effects of localized mixing over topography. Effects of realistic topography have been examined in the North Atlantic by Karcher and Lippert (1994) and the flow of Antarctic Bottom Water in the Atlantic by Stephens and Marshall (2001). The flow is found to be essentially along geostrophic contours and is characterized by strong topographic slope currents and interior recirculation.

In the present paper we present a linear analytical treatment of the effect of bottom topography on the steady state circulation of Stommel’s (1958) model. We retain the uniform upwelling in the interior as assumed by Stommel and study the role of sloping boundaries on the redistribution of the water that sinks in the northeastern corner of a basin in the northern hemisphere. The oceanic interior has uniform depth but the bottom shoals linearly to the surface near the west and north and there is a mid-ocean ridge running north-south and with sides extending upward linearly three kilometers from a uniform depth of four kilometers (Fig. 1).

Our intent differs from preceding ones in that we treat the entire two-layer problem including the sinking as well as the upwelling regions. Our study is a gfd type of exploration intended to understand processes and behaviors rather than to simulate nature. The upwelling is uniformly distributed where the depth is constant but we have neglected the forcing over the slope regions which are assumed to be narrow compared to the constant-depth interior. (This has been standard procedure for calculations of wind-driven circulation and leads to results that are negligibly different from ones where the forcing is kept in narrow boundary layers.) We have obtained an exact analytical solution to the asymptotically inviscid, linear problem for both the homogenous and the two-layer model. By asymptotically inviscid we mean that we have obtained those aspects of the flow that can be derived without specifically taking into account the effect of friction. However, we have been forced to treat regions of flow that do not merge smoothly with adjacent regions. By doing so, we have identified a number of discontinuities around which there must be boundary layers in which friction is important and turbulent dissipation can take place. We have been guided, in part, by an analysis of Salmon’s (1992) in which he treated the wind-driven circulation of a two-layer system. In our case, however, there are isolated jets embedded in the middle of the discontinuities. Therefore, a numerical treatment to take friction into account
will be necessary if the different regions are to be connected smoothly. That is the aim of a later study. To obtain an analytical solution it was necessary to have the depth of the upper layer vanish along the eastern boundary south of the north slope.

Recent research indicates that part of the vertical transport in the deep ocean is due to enhanced mixing at turbulent locations over rough topography (Sjöberg and Stigebrandt, 1992; Ledwell et al., 2000; Gustafsson, 2001; Spall, 2001). Such a possibility has, in fact, been the subject of speculation ever since Stommel (1958) assumed uniform upwelling. In our analytical treatment we are pursuing the logical consequences of topographic influence on Stommel’s model and we must leave the subject of the influence of local topographic anomalies to a later, numerical study.

In the following section we first present the general equations governing a two-layer flow. The case of a single abyssal layer, i.e. Stommel’s problem with variable depth, is treated in section 3. In section 4 we outline the solution for the two-layer interior ocean where the depth is constant and the upwelling across the interface is uniform. We review some general theorems that Salmon (1992) obtained for a two-layer ocean with topography and then apply the results of those theorems to obtain the solution to the thermohaline circulation driven by uniform upwelling.

2 The model

We will study a model of a single hemispheric basin. The effect of stratification is represented, in its simplest form, by two layers with density \( \rho_1 \) in the upper layer and \( \rho_2 \) in the lower layer. The flow will be driven by uniform upwelling \( w_o \) through the interface between the layers. The thickness of the layers is \( h_1 \) and \( h_2 \) respectively. The height to the surface from the flat ocean floor is \( \eta_1 \) and the height to the interface is \( \eta_2 \). The ocean floor is flat in the interior of the basin but in regions narrow compared to the width of the basin the bottom topography varies in such a way that we have a symmetric ocean ridge in mid basin and sloping walls at the northern and western coasts (Fig. 1). The flow is steady and hydrostatic and each layer is in geostrophic balance. In the linear theory, friction will become important only in thin frontal regions. We use a simple linear drag for the frictional terms given by \( -Ru_i \) where \( R \) is constant and \( u_i = (u_i, v_i) \) is the velocity in the eastward (x) and northward (y) directions. Subscript 1 is used for the upper-layer velocities and 2 for the lower layer. The equations of motion for this system take the following form
\[ f \times u_1 = -g \nabla \eta_1 - R u_1 \]  
\[ f \times u_2 = -g \frac{\rho_1}{\rho_2} \nabla \eta_1 - g \frac{\Delta \rho}{\rho_2} \nabla \eta_2 - R u_2 \]  
\[ \nabla \cdot (u_1 h_1) = w_0 \]  
\[ \nabla \cdot (u_2 h_2) = -\frac{\rho_1}{\rho_2} w_0 . \]  

A \( \beta \)-plane approximation will be used so that the Coriolis parameter \( f \) is \( \beta y k \). Following the outline of Salmon (1992) the above equations lead to equations for the transport streamfunction and the upper layer thickness of the system.

The transport streamfunction \( \Psi \) for the vertically integrated flow is defined as

\[ \frac{\rho_1}{\rho_2} h_1 u_1 + h_2 u_2 = k \times \nabla \Psi . \]  

Introducing the reduced gravity \( g' \equiv \frac{(\rho_2 - \rho_1) g}{\rho_2} \) where \( g \) is the gravitational acceleration and as \( \eta_2 = \eta_1 - h_1 \) the upper- and lower layer velocities can be written in the form

\[ \frac{\rho_1}{\rho_2} u_1 = \frac{1}{H} k \times \nabla \Psi + \frac{g' h_2}{f H} k \times \nabla h_1 - R \frac{g' h_2}{f^2 H} \nabla h_1 \]  
\[ u_2 = \frac{1}{H} k \times \nabla \Psi - \frac{g' h_1}{f H} k \times \nabla h_1 + R \frac{g' h_1}{f^2 H} \nabla h_1 \]  

where \( H = h_1 + h_2 \) is the total depth and terms of order \( R^2 \) \( (R \ll f) \) have been neglected.

We can substitute the expression for \( u_1 \) into (3) and this leads to the equation for the upper-layer thickness \( h_1 \)

\[ J \left( \frac{\Psi, h_1}{H} \right) + J \left( g' h_1, \frac{h_1}{f} \left( 1 - \frac{h_1}{H} \right) \right) = \nabla \cdot \left( R \frac{g' h_1}{f^2} \left( 1 - \frac{h_1}{H} \right) \nabla h_1 \right) + \frac{\rho_1}{\rho_2} w_0 . \]  

The horizontal Jacobian \( J \) is defined as \( J(A, B) \equiv (A_x B_y - B_x A_y) \) where subscripts denote differentiation.
The equation for the streamfunction is obtained from the curl of the vertically averaged momentum equation

\[ J(\frac{f \cdot H}{\Psi}) + J(\frac{g' h^2}{2 \cdot H^2}) = \nabla \cdot \left( \frac{R}{H} \nabla \Psi \right) \quad (9) \]

### 3 The homogeneous model

We will first consider the case in which the ocean is represented by a single active layer under an infinitely deep and motionless upper layer. The bottom topography of the basin is shown in Fig. 1. The 500 km wide western and northern boundaries slope linearly from the ocean surface to the flat ocean floor at a depth of 4000 m. The mid-ocean ridge has a height of 3000 m and is 1000 km wide. The southern boundary is located at the equator.

![Fig. 1. The topography of the model basin.](image)

The flow in the basin is driven by uniform upwelling, \( w_o \), through the upper surface. The upwelling takes place over the flat bottom but is neglected over the sloping boundaries. As stated earlier, neglecting the forcing in very thin boundary layers involves negligible error. The slope regions are assumed to be narrow compared to the interior basin width; also, because of much higher velocities over the slopes, upwelling in these regions, which induces a small cross-PV velocity, would have little effect on the flow. The layer gains water from inflow through the wall at the north-eastern corner where the source
transport is proportional to depth across the northern slope at that corner. We will first turn to the solution for the interior basin. This solution will then be matched to the sloping boundary solutions.

3.1 Flat-bottom interior

The set-up of the interior is the same as in Stommel’s (1958) abyssal circulation model. The velocity components read

\[ v = w_0 \frac{y}{h} \]  
\[ u = \frac{2w_0}{h} (x_E - x) \]

where \( w_0 \) is the upwelling velocity and \( u \) vanishes at \( x = x_E \). To get these linear results we have neglected the deviation of the height thickness, \( \eta \), compared to \( h \).

Trajectories of the flow are determined by

\[ \frac{u}{v} = \frac{dx}{dy} = 2 \frac{(x_E - x)}{y} \]

so integrating from \((x_W, y_W)\), the point at the foot of the western slope from which the trajectory enters the interior, to \((x, y)\) yields the following equation for the trajectory in the interior

\[ x = (x_W - x_E) \frac{y_W^2}{y^2} + x_E \]

3.2 Slope-regions

With the assumption that there is no upwelling in the slope regions we can define a transport streamfunction \( \Psi \) by \( hv = \Psi_x \) and \(-hu = \Psi_y \) and the conservation of PV over the slopes becomes

\[ J \left( \frac{f}{h}, \Psi \right) = 0 \]
As one expects in the undriven flow, the transport streamfunction is constant on lines of constant $f/h$ as a fluid column must conserve its potential vorticity (PV). In order for the flow to move northward or southward, it must hence also move downslope or upslope.

3.3 Results for the homogenous case

With upwelling neglected in the sloping boundaries and over the mid-ocean ridge, the flow over these regions is, in the zero-order picture, driven only by matching to the flow in the flat-bottom interior of the basin. Upwelling over the slope would generate flow perpendicular to lines of constant $f/h$ in the upslope direction, so neglecting $w_0$ means that there is a small, essentially negligible, geometric distortion in the flow pattern over the slope. In order to match the solution for the interior to the slopes, the flow must conserve the PV it has at the foot of the slope when flowing into the boundary regions. With this matching condition, we obtain the flow pattern shown in Fig. 2.

In the flat-bottom interior of the basin we have the flow-pattern proposed by Stommel (1958). The path of the flow is everywhere toward the north as a result of the upwelling-induced vertical stretching of fluid columns. The zonal flow vanishes at the eastern boundary.

Fig. 2. Trajectories of the flow
When interior flow intersects the mid-ocean ridge, it must conserve the PV it has at the foot of the slope. To climb the slope it must thus turn southward as the depth of the fluid column decreases. At the top of the ridge, the flow turns back northward as the depth of the fluid column starts to increase and it matches to the interior at the same latitude as when it first encountered the ridge. The result is a V-shaped structure symmetric about the ridge. These trajectories on the ridge are dashed.

When interior flow reaches the northern boundary it will have to flow westward along the foot of the slope. The part of the interior flow that crossed the ridge during its northeastward path, will again have to cross the ridge during its westward path. It will undergo the same V-shaped path across the ridge, first southward and upslope and then northward and downslope before it joins to the jet at the foot of the northern slope on the western side of the ridge, as shown by the solid V-shaped contour.

All of the interior flow along the foot of the slope will have to come southward in the western boundary layer on the line of constant $f/h$ that runs across this boundary, from the north-western corner of the interior basin to the south-western corner of the western boundary layer. To the right hand side of this $f/h$ line, fluid flows northward on lines of constant PV to match anew to the interior flow. On the left hand side of the line, source water from the north is flowing southward.

The source water comes in to the northern boundary through the eastern wall of the northern slope. Shallower columns of water flow westward on the isobath of entry. Deeper columns encounter the mid-ocean ridge and are diverted into the V-shaped pattern to cross the ridge. In the western boundary layer the flow turns southward and comes upslope on lines of constant PV, set by the fluid depth in the northern boundary.

In this asymptotically inviscid system, it is clear that friction will have to act in certain regions. At the foot of the northern boundary we have a westward jet that contains all of the flow from the interior. This infinitesimally thin jet has to be smoothed out on both sides to adjoin the flow in the interior and on the slope. Friction will also have to act in the vicinity of the jets on the slopes and on the ridge as well as along the top of the ridge. In a narrow region at the south wall of the western slope, friction will be needed to enable the southward flow to cross lines of constant PV and turn northwards to join on to the trajectories of the interior basin.

In order to check on the linearization assumption for this flow (also needed in section 4) we examined the magnitude of $\Delta \eta$ relative to $h$ by calculating $\Delta \eta$ at a point on the western slope where $y = y_N/2$. Since this is a reduced gravity model, the departure of the interface from equilibrium makes use of $g'$ and
is therefore also a measure of what can be expected in the more complicated two-layer system of the next section.

Multiplying the geostrophic equation with $h$ leads to $fhw = f\Psi_x = g'\eta_x$, which can be integrated horizontally along $y_{N/2}$ from the foot of the western slope region, $x_I$, to a point $x$ halfway up the slope, to give $\Delta \eta$, the difference in $\eta$ between those two points. In order to evaluate $\Psi_x$ for this integration, we note that the values of $\Psi$ along $x_I$ (obtained from the interior values of $uh$) are linear in $y$, and from any value of $y$ northward of $y_{N/2}$ the projection is along $f/h$ lines onto the horizontal line $x_I$ to $x$ at $y_{N/2}$. Therefore, $\Psi$ is a linear function of $x$ so $\Psi_x$ is a constant. On the slope the thickness of the layer is $h = 4x/500 \equiv sx$, so we have $\eta_x = B/x$, where $B = f\Psi_x/(g's)$. An integration leads to $\Delta \eta = B \ln(x/x_I) = B \ln(1/2)$ halfway up the western slope. With $\beta = 2 \cdot 10^{-11}$ (m s), $g' = 0.015$ m/s$^2$ and $w_0 = 2 \cdot 10^{-7}$ m/s this gives $\Delta \eta = 13.7$ m, which is only about 0.3 % of the fluid depth, so the linear approximation involves a negligible error. In the two-layer case $\Delta \eta$ must be compared to the depth of the upper layer which is about 200 m (see Fig. 5), so the error is 6.8 %.

Numerical studies of abyssal flow over topography confirm the pattern seen in Fig. 2. Condie and Kawase (1992) used a one and a half layer model to represent the abyssal flow over exponentially sloping side walls at all boundaries and a Gaussian shaped mid-ocean ridge. Their general flow pattern closely parallels geostrophic contours i.e., the fluid flows nearly along lines of constant $f/h$ and the same (smoothed) V-shaped structure of the flow is visible at the mid-ocean ridge.

Some numerical support for the present approach is provided by Salmon (1992), who carried out a numerical integration of the equations for homogeneous fluid driven by a double gyre wind stress pattern over a single hemispheric basin with uniform continental slopes at the boundaries. He first analyzed the system as we have done (we were guided by his calculation) and then his numerical study showed how friction rounded the corners and allowed for a small departure from the trajectories of the inviscid system.

One aspect of Salmon’s study that is reflected in the observations is the penetration of water from the subpolar gyre into the western boundary slope region of the subtropical gyre. The observations in the Slope Sea reported by Csanady and Hamilton (1988) show just that kind of southward penetration of subpolar water with a cyclonic circulation inshore of the Gulf Stream (Fig. 3), although that circulation may also be influenced by the northern gyre of recirculation associated with the Gulf Stream. The depth of the water in the Slope Sea is sufficiently small that Salmon’s barotropic analysis may be pertinent.
We cannot use Salmon’s (1992) study for complete justification of our model because the wind-driven circulation does not contain the discontinuity provided by the southwestward jet over the slope that our upwelling model does so there is still an aspect of our calculation that requires numerical confirmation. We are working to supply that confirmation.

4 The two-layer model

In section 3, a geometrical picture was obtained of how the influence of sloping topography will alter the boundary layer flow of a homogeneous fluid. Extending the theory to a baroclinic flow, we will seek the solution to the general equations (8) and (9). In the inviscid limit they become

\[ J \left( \Psi, \frac{h_1}{H} \right) + J \left( g' h_1, \frac{h_1}{f} \left( 1 - \frac{h_1}{H} \right) \right) = \frac{\rho_1}{\rho_2} w_o \]  

(15)

and

\[ J \left( \frac{f}{H} \Psi \right) + J \left( g' \frac{h_1^2}{2}, \frac{1}{H} \right) = 0. \]  

(16)

Again we treat the interior and the sloping boundaries separately and match the solutions at the foot of the slopes.
4.1 Two-layer interior

We make use of the results of Veronis (1976). In the flat bottom interior the total depth $H$ is constant so for this flow driven by upwelling there is no barotropic flow, i.e.,

$$\Psi \equiv 0$$  \hspace{1cm} (17)

everywhere in the interior. Hence, from (15)

$$\frac{-g'\beta}{f^2} \left( \frac{h_1^2}{2} - \frac{h_1^3}{3H} \right) = \frac{\rho_1}{\rho_2} w_0 \cdot$$  \hspace{1cm} (18)

Integrating (18) from $x$ to the eastern boundary $x_E$, we can write the expression for the upper-layer thickness

$$\frac{g'\beta}{2f^2} \left( h_1^2 \left( 1 - \frac{2h_1}{3H} \right) - h_1^2 \left( 1 - \frac{2h_{1E}}{3H} \right) \right) = \frac{\rho_1}{\rho_2} w_0 (x_E - x)$$  \hspace{1cm} (19)

where $h_{1E}$ is the specified upper-layer thickness at the eastern boundary. With $h_2 = H - h_1$ we can now also calculate the thickness of the lower layer.

The curl of the vertically averaged momentum equations for each layer yields the vertically integrated meridional transports

$$V_1 = -\frac{fw_0}{\beta} \equiv -yw_0 $$  \hspace{1cm} (20)

$$V_2 = \frac{\rho_1}{\rho_2} \frac{fw_0}{\beta} \equiv \frac{\rho_1}{\rho_2} yw_0 .$$  \hspace{1cm} (21)

The meridional transports are equal and opposite at every point in the respective layers, always southward in the upper layer and northward in the lower layer.

The vertically integrated zonal transports can be derived by substituting (20) and (21) into the continuity equations (3) and (4) and integrating from $x$ to the eastern boundary $x_E$.
\begin{align*}
U_1 &= -2w_0(x_E - x) \\
U_2 &= \frac{\rho_1}{\rho_2} 2w_0(x_E - x)
\end{align*}

where \( U_i(x_E) \) vanishes.

### 4.2 Two-layer model with topography

Having obtained a solution for the flat bottom interior, we will seek a solution for a two-layer system with the bottom topography shown in Fig. 1. As in the homogenous case we neglect the upwelling in the sloping regions. So, for the inviscid case we again seek solutions to equations (15) and (16), allowing for \( H \) to vary and with \( w_0 = 0 \). We will here make use of the general solutions obtained by Salmon (1992). He showed that by defining

\begin{align*}
q_1 &
\equiv \frac{h_1}{f}, \quad q_2 \equiv \frac{H - h_1}{f} \equiv \frac{h_2}{f}
\end{align*}

\( \text{(24)} \)

equations (15) and (16) (with \( w_0 = 0 \)) can be written in the forms

\begin{align*}
J(\Psi, q_1) + q_2 J(g'h_1, q_1) &= 0 \\
J(\Psi, q_2) - q_1 J(g'h_1, q_2) &= 0
\end{align*}

\( \text{(25)} \)

\( \text{(26)} \)

which describe constraints on PV in the upper and lower layers.

The general solutions that Salmon (1992) obtained were for the two cases \( J(q_1, q_2) \neq 0 \) and \( J(q_1, q_2) = 0 \). In the latter case he showed that a solution is possible only if either \( q_1 \) or \( q_2 \) is constant in a particular region. That would appear to be simply a mathematical curiosity but he showed that, in fact, one can obtain realistic features in the vicinity of the Gulf Stream by making use of \( q_1 = \text{constant} \) and \( q_2 = \text{constant} \) in different regions near the Gulf Stream.

We have sought a solution with \( q_1 = \text{constant} \) over the western slope. In order for that to be a valid solution it is necessary that it match to the interior solution at the foot of the slope. The latter comes from \( \text{(19)} \). In the special case with \( h_{1E} = 0 \), we can obtain a solution with constant \( q_1 \) at the foot of the slope if we take the lowest order solution to \( \text{(19)} \) by treating the term with \( H \) in the denominator as constant in the flat bottom interior. Take \( K = (1 - \frac{2h_1}{3H}) \),
where $h_1$ is the average upper layer thickness in the interior. The resulting solution is

$$
\frac{h_1^2}{f} = \frac{2f^2 \rho_1 w_0}{Kg' \rho_2 \beta} (x_E - x) .
$$

(Eq. 27)

Evaluating (27) at the western edge of the interior, i.e. $x = x_W$, we have

$$
\frac{h_1}{f} = \left( \frac{2\rho_1 w_0}{Kg' \rho_2 \beta} (x_E - x_W) \right)^{\frac{1}{2}}.
$$

(Eq. 28)

So $q_1 \equiv h_1/f = constant$ is an exact solution over the western boundary slope and it also matches the interior at the eastern edge of the boundary. Therefore, as $f = \beta y$, $h_1 = constant \times f$ is constant on lines of constant $y$ in the boundary layer. The matching condition is that the value of $h_1(y)$ in the boundary layer is set by the thickness of the interior upper layer at $x = x_W$ as given by (28). The largest error produced by the assumption of $h_1/H \approx \frac{h_1}{H}$ occurs at the northwestern corner of the interior and involves a maximum error there of about 1.6%.

In the same way, (27) can be evaluated at the foot of the mid-ocean ridge to give $h_1/f = constant$ in the ridge area. The upper-layer thickness across the ridge is constant for each $y$ and determined by the upper-layer thickness where the slope meets the interior at that value of $y$.

In the absence of upwelling over the slopes the PV of the lower layer is conserved and so we have

$$
J \left( \Psi_2, \frac{f}{h_2} \right) = 0 .
$$

(Eq. 29)

So in the lower layer, the transport streamfunction is constant on lines of constant $f/h_2$. As we know that $h_1$ is constant on lines of constant $y$ in the slope regions, $h_2$ can be determined from $h_2 = H - h_1$.

4.3 Transport balances

In the classical study of the abyssal flow, dissipative western boundary regions of thickness $R/\beta$ are appended to the flow in order to conserve mass. We will also here use transport balances in the argumentation for the proposed flow in our two-layer basin, so we proceed in this section to review some transport
properties. In calculating the transports, we neglect the presence of the mid ocean ridge, assuming that it has infinitesimal width. We used 1000 km for the width only for visual convenience.

In the steady state, conservation of mass requires that the total transport in the respective layers vanishes, i.e.

\[
T_1 + T_{1W} + W + S = 0 \tag{30}
\]

\[
T_2 + T_{2W} + \frac{\rho_1}{\rho_2}W + \frac{\rho_1}{\rho_2}S = 0 \tag{31}
\]

Here \(T_1\) and \(T_2\) are volume transports in the interior and \(T_{1W}\) and \(T_{2W}\) are volume transports in the western boundary currents. \(W\) is the volume of fluid that upwells through the interface and \(S\) is the volume that sinks at the northeast corner.

The zonally integrated interior transport in the upper layer, \(T_1\), across a zonal line \(y\) is given by integrating \(V_1\) over the basin width

\[
T_1 = -w_0(x_E - x_W)y \tag{32}
\]

The upper layer gains water from the total upwelling taking place north of \(y\). This is given by integrating \(w_0\) over the area of the basin north of \(y\).

\[
W_1 = w_0(x_E - x_W)(y_N - y) \tag{33}
\]

The net loss of upper-layer fluid to the lower layer at the northeastern corner must equal the total amount of fluid upwelled so we have

\[
S_1 = -w_0(x_E - x_W)y_N \tag{34}
\]

Equations (30) to (34) therefore determine the transport \(T_{1W}\) carried in the western boundary current at a zonal line \(y\)

\[
T_{1W} = 2w_0(x_E - x_W)y \tag{35}
\]

From (35) it can be seen that the western boundary current transports twice the amount of fluid as the interior transport across each zonal line \(y\) but in the opposite direction. In the lower layer the transports are equal, except for
the factor $\rho_1/\rho_2$, and opposite to those of the upper layer. The distributions of the transports in the slope regions are given next.

4.4 Results for the two-layer model

The zero-order flow pattern for the lower layer is shown in Fig. 4 and for the upper layer in Fig. 5. The dashed lines in the flat bottom interior are the trajectories of the flow and the solid lines are contours of the height of the interface above the flat bottom in Fig. 4 and the thickness of the upper layer in Fig. 5. The calculation was done for a vanishing upper-layer thickness at the eastern boundary and with a flat bottom basin depth of 4000 m. The upwelling velocity was specified to $2 \cdot 10^{-7}$ m/s and $\Delta \rho/\rho_2 = 0.0015$.

In the interior of the basin the solution is that given in section 4.1, with the contours of the layer thicknesses parallel to the trajectories of the flow.

Over the sloping boundaries of the lower layer, topography influences the flow in the same manner as in the homogenous case. A fluid column has to conserve the PV it has at the foot of the slope and will hence flow on lines of constant

Fig. 4. Flow pattern of the lower layer (dashed lines) with values of the height of the interface above the flat bottom given in meters on the solid curves. The calculations were made for $w_0 = 2 \cdot 10^{-7}$ m/s, $h_E = 0$ m and $\Delta \rho/\rho_2 = 0.0015$
Fig. 5. Flow pattern of the upper layer (dashed lines) with values of upper layer thickness $h_1$ on the solid contours. The dotted line over the ridge indicates the location of the lower layer jet.

$h_2/f$. Also in the two-layer case fluid from the interior produces a westward jet along the foot of the northern slope. Over the western boundary slope this jet turns southward along the $h_2/f$-line that originates at the northwest corner of the interior and runs to the southwest corner of the western boundary. Fluid to the west and east of the jet flows along $h_2/f$-contours, southward on the western side and northward on the eastern side as shown in Fig. 4.

In the upper layer the flow over the western boundary slope and over the mid-ocean ridge is strictly zonal on lines of constant $h_1/f$ due to the constant upper-layer thickness in these regions (Fig. 5).

The upper layer in the western boundary needs to transport the same amount of fluid northward as flows southward in the lower layer. Just south of the southern edge of the northern boundary slope, the upper-layer western boundary transport is

$$T_{1W} = S_1 + T_1 .$$  

(36)

Just north of that edge, the interior transport is zero, which gives
Hence, there must be a jet that heads eastward along the foot of the northern slope in the upper layer. This jet will supply the interior of the basin with fluid. Upper-layer flow over the western boundary slope must therefore deliver the fluid needed in the interior to the northwestern corner of the interior in order to supply the eastward flow along the foot of the northern slope. So there must be a jet in the western boundary upper-layer flow. This jet will carry the same amount of fluid northward as the jet in the lower layer does southward along the line of constant $h_2/f$.

In that jet, to balance the northward transport in the upper layer, the sea surface will rise sharply toward the east. We must also have a compensating drop of the interface. The displacement of the interface across the western boundary jet can be obtained from the transport equations for the upper and lower layer, (1) and (2), with $R = 0$

\begin{align}
\rho_1 fV_1 &= \rho_1 g h_1 (\eta_1)_x \\
\rho_2 fV_2 &= gh_2 (\rho_1 \eta_1 + \Delta \rho \eta_2)_x .
\end{align}  

(38)  

(39)

As the mass transports in the upper and lower layer are equal and opposite, the total transport vanishes at each point also in the jet. From (38) and (39) we then get

\begin{align}
(\eta_2)_x &= -\frac{\rho_1}{\Delta \rho h_2} \frac{H}{h} (\eta_1)_x .
\end{align}  

(40)

As $h_1 = \eta_1 + h - \eta_2$, where $h$ is the mean upper-layer depth of the jet, we have

\begin{align}
(\eta_2)_x = (\eta_1)_x - (h_1)_x .
\end{align}  

(41)

Applying (41) to (40) we find

\begin{align}
(\eta_1)_x &\approx \frac{\Delta \rho h_2}{\rho_1 H} (h_1)_x
\end{align}  

(42)

and (38) becomes

\begin{align}
fV_1 &= g \frac{\Delta \rho h_2}{\rho_1 2H} (h_1^2)_x .
\end{align}  

(43)

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Since the assumption is of an infinitesimally thin jet, we can consider \( h_2/H \) to be constant when we integrate with respect to \( x \) across the jet so that (43) becomes

\[
fT_1 = g \frac{\Delta \rho}{\rho_1} \frac{h_2}{2H} \left( h_{1R}^2 - h_{1L}^2 \right)
\]  

where subscripts R and L denote the right and left edge of the jet respectively.

\( h_1/f \) is constant across the western boundary layer, from the interior to the jet. At the right edge of the jet the upper-layer thickness is thus the same as the upper-layer thickness at the edge of the interior, i.e \( h_{1W} = h_{1R} \). So we can obtain the equation for the upper-layer thickness at the western side of the jet

\[
h_{1L}^2 = h_{1W}^2 - 2f \frac{\rho_1 H}{g \Delta \rho h_2} T_1.
\]  

Now (45) holds along \( h_2/f = constant \) and along that line \( H \) varies linearly from 4000 m at the northeastern corner of the slope to 0 at the southwestern corner so \( H \propto y \). Since \( T_1 \propto y \) from (32) and \( h_{1W} \propto y \), we have \( h_{1L}^2 \propto y^2 \) so \( h_1/f \) is constant on the western side of the jet and the flow is purely zonal.

Close to the western edge of the boundary layer, there will be a triangular wedge of only upper-layer water, where the transport is northward as the streamfunction must go to zero at the western edge of the boundary layer. This northward transport supplies fluid to the sinking region.

M. Kawase pointed out to us that the lower layer jet over the mid-ocean ridge must also involve a change in the depth of the interface and a corresponding change in the thickness of the upper layer. In other words, our assumption that \( h_1 \) is constant over the ridge is not correct and the change of depth implies a change in PV which means that friction must play a role in the dynamics of the upper layer. We calculate the change in thickness of the upper layer induced by the lower layer jet as follows:

All of the lower layer water that impinges on the foot of the north slope must be transported by the jet over the ridge. Multiplying (21) by 3000 km, the width of the eastern basin, gives a total transport of \( T_2 = 3.9 \cdot 10^6 \) m\(^3\)/s. We can assume reduced gravity dynamics for the jet since the upper layer flow is comparatively weak, so integrating the geostrophic equation across the jet yields the change in the interface height, \( \Delta \eta = fT_2/g'h_2 \). Evaluating this expression leads to \( \Delta \eta = 9.1 \) m over the ridge. This is about 3% of the upper layer thickness at \( y = 6500 \) km. Therefore, it is clear that the flow in the vicinity of the jet will be influenced by friction in both the upper and the
lower layers. How the eastward upper-layer jet along $y = 6500$ km crosses the ridge is not uniquely determined by our solution. It could parallel the lower layer jet but flow in the opposite direction. Or it could continue to flow eastward across the ridge since the upper layer lies well above the ridge. We have indicated the second solution in Figs. 5 and 6. In either case, friction will have an influence where the jump in the upper layer thickness occurs.

A schematic of the cross-sections in the western boundary, the ridge area and the northern boundary is shown in Fig. 6. In the upper layer at the western boundary (Fig. 6a) there is a westward transport from the foot of the slope towards the jet. In the jet the transport is northward and carries fluid to the jet along the northern boundary that supplies the interior with water. The upper-layer thickness decreases across the jet. On the western side of the jet the upper-layer thickness is again constant and carries water westward to the northward flowing boundary current that supplies the sinking region.

In the lower layer, the region closest to the western edge of the boundary layer carries source water southwards from the northern boundary. The southward moving jet in the lower layer carries water from the westward jet of interior water that runs along the foot of the northern slope. On the eastern side of the jet, a northward flow supplies this water and the source water back to the interior.

Over the mid-ocean ridge the flow is westward (Fig. 6b) and $h_1/f$ is constant.
except for a jump in the upper layer thickness across the lower layer jet. In the lower layer, source water flows southward in the eastern part of the ridge and northward on the western part. In the two outer regions of the ridge, fluid from the western interior crosses the ridge. At the intersection between the regions, the jet along the foot of the northern slope turns southward on the east part of the ridge and northward on the west part of the ridge.

On the northern slope (Fig. 6c) the upper-layer jet at the southern edge of the boundary layer supplies fluid to the interior while the current at the northern edge of the boundary layer provides water to the sink. The transport of water to the sinking regions must spread out across the upper layer, but the details of this flow have not been analyzed. In the lower-layer jet, water from the interior is carried westward, and north of the jet the source water is carried to the western boundary layer.

5 Summary and discussion

A single basin in the northern hemisphere with two layers, driven by uniform upwelling through the interface of the layers, has been treated analytically for asymptotically small friction. The solution to the flat bottom ocean by Veronis (1976) has been extended to include slopes at the western and northern boundaries and also a mid-ocean ridge. We have obtained a simple, analytical solution consistent with an exact solution to the general equations (15) and (16) for the case when \( J(q_1, q_2) = 0 \).

In the presence of a mid-ocean ridge dividing the ocean into two halves, the flow in the lower layer must make a deep meridional excursion to cross the ridge. The flow pattern over the mid-ocean ridge of our model resembles that in Fig. 7 which shows the deep circulation estimated from observations by Defant (1941, 1961). The similarity in the deep flow patterns over the ridge between theory and observations may provide some support for Stommel’s assumption that the deep flow is generated by upwelling.

The resulting flow over the slope regions is along lines of constant \( h_1/f \) in the upper layer and \( h_2/f \) in the lower layer in order to conserve the PV that it has at the foot of the slope. The solution requires a northward jet in the upper layer of the western boundary layer, from the southwest corner of the basin to the northwest corner of the interior, which continues eastward along the foot of the north slope. Friction is clearly needed in the vicinity of all jets to smooth out the flow, and also in a thin layer close to the equator in the western boundary layer in order to enable the flow to cross potential vorticity contours. Although friction must be important in parts of the slopes, we have confined the analysis to the inviscid limit using transport balance arguments.
This led to the conclusion that half of the total northward transport of upper-layer water in the western boundary layer will take place in the jet and half will occur in a triangular wedge occupied by only upper-layer water along the western edge of the western slope. In the real ocean the wind driven circulation is largely confined to the upper layer because of the stratification and there is little evidence of topographic influence. In this model, in which the flows in both layers are due to the same driving force, it is particularly significant that the upper layer flow is independent of the shape of the ridge.

In order to obtain our solution we had to assume that the upper-layer thickness vanishes on the eastern boundary. That is not a serious limitation for the pure thermohaline circulation problem. It may pose more of a problem for the combined case with wind forcing where a deep bowl of upper-layer water occurs in the western basin.

The exact dynamics over the northern slope have not been analyzed, but the lower-layer flow follows lines of constant $H$ (total depth). Close to the northern edge of the north slope we again have dynamics of a single layer as the total
depth goes to zero. Upper-layer water flows out near the northeastern corner and the resulting lower-layer transport in the northern boundary region flows westward and then southward along lines of constant $f/h_2$ to the southwest corner of the western boundary slope.

The analytical solution was pieced together, treating the slope regions in the inviscid limits. The reasoning holds for slope regions that are very narrow compared to the basin size. The jets in each layer represent lateral discontinuities in the normal derivatives of the streamfunction on both sides of the jets. This issue can be studied in the single layer problem since the jet occurs there as well, so a numerical analysis of that problem should serve to clarify how friction enables the jet to merge with the adjoining flows. It is by no means obvious that friction simply provides a boundary layer appendage to the inviscid solution. It is quite possible that friction will introduce an element not even hinted at in our analysis. So we feel that a numerical solution is a necessary supplement to the present one.

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